Wiener crosses borders: interpolation based on second order models

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ABSTRACT

Interpolation of signals (arbitrary dimension, here: 2D images) with missing data points is addressed from a statistical point of view. We present a general framework for which a Wiener-style MMSE estimator can be seamlessly adapted to deal with problems such as image interpolation (inpainting), reconstruction from sparse samples, and image extrapolation. The proposed method gives a precise answer on a) how arbitrary can linear filters can be applied to initially incomplete signals and b) shows the definite way to extend images beyond theirs borders such that no size reduction occurs if a linear filter/operator is to be applied to the image.

Keywords: Linear statistical signal model, Wiener filter, least squares prediction, inpainting, extrapolation

1. INTRODUCTION

The problem of reconstructing a signal from partial or missing data has been investigated extensively in the literature. In particular, data interpolation and extrapolation has found many more applications in image processing than can be listed here; very current ones being e.g. in face recognition and in object tracking from occluded data.

In this paper we devise an efficient method to optimally (in a least squares sense) reconstruct incomplete discrete signals, including the case of image interpolation, filling “holes” (a.k.a. “inpainting”), and extrapolation. The central assumption is that the signal can be sufficiently well modeled by a stationary process with known first and second order statistics. We strongly agree with Meijering in the observation that statistical approaches to interpolation have received little attention in the image processing literature. The present contribution aims at reversing this situation.

The mathematical core of this method is essentially a direct consequence of Wiener’s theory developed in the 1940ies; in the image and signal processing communities a surprisingly widespread misconception can be found that this induces using Fourier space modelling. Strange enough, more often than in signal processing we find statistical reconstruction theory to be used in geostatistics due to the work of Krige and Matheron, which builds a rich, widely used method repertoire on variants of Wiener’s LS approaches. Besides its direct relation to ‘Kriging’, our approach extends the work of Leung et al., which only deals with interpolation of regularly subsampled images.

We claim that the approaches given here provide a fresh and insightful view on the problem of reconstructing image with missing data. Particularly for the problem of extrapolating an image, we are not aware of any specific work following these lines.

The proposed reconstruction formula has the flexibility to not only interpolate/extrapolate a given signal with missing data, but also to seamlessly handle any linear filtering operation performed on the signal. This gives a definite answer to the problem of how an image should be extended across its borders if an arbitrary filter has to be applied to it, without reducing the size of the resulting image.

Another feature of the present approach is its ability to adapt the interpolation to the particular and individual structure of the image signal; be it either globally for a class of images, or individual for a single image, or even locally adaptive. The only information required is an estimate of the autocorrelation function (ACF).

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The proposed method allows also the statistically correct processing of noisy samples, but in contrast with the well-known smoothing spline approach, all the required intrinsic parameters are directly dependent on the observable data and the process statistics. Furthermore, the extrapolation scheme described here can use, but does not depend on an elaborate choice of a basis function representation, as found for instance in.

2. SIGNAL MODEL

In this section we present the mathematical derivation of the model to be used for the interpolation and extrapolation tasks. We devise a linear model for the reconstruction of missing data, in the same line as performed in. We derive the model for vectors, as higher dimensional data can always be arranged as a large stream of uni-dimensional data.

We assume that the input signal vector \( \vec{s} \in \mathbb{R}^M \) has an observable part \( \vec{z} \in \mathbb{R}^N \), with \( N < M \) and the remaining dimensions are not observable, or missing. The measurable vector \( \vec{z} \) can be expressed as

\[
\vec{z} = \vec{K} \cdot \vec{s}
\]

where \( \vec{K} \) is a \( N \times M \) matrix called the observable matrix. From a mathematical standpoint there are no special requirements on \( \vec{K} \); however, in the case of missing data, \( \vec{K} \) can be simply taken to be a matrix of only zeros and ones, according to the observability of the values of \( \vec{s} \). A more realistic model would incorporate a noise term \( \vec{v} \) to the measurement vector \( \vec{z} \) as follows

\[
\vec{z} = \vec{K} \cdot \vec{s} + \vec{v}.
\]

Our goal is then to find an interpolation formula to fill in (=estimate) the missing values in \( \vec{s} \). The estimation of \( \vec{s} \) can be performed element by element; the goal is always to estimate the value \( g = \vec{h}^T \cdot \vec{s} \), where \( \vec{h} \in \mathbb{R}^M \) can be a trivial vector \((0,0,\ldots,1,\ldots,0)\) if the complete vector \( \vec{s} \) is to be estimated sample by sample. But \( \vec{h} \) can be any linear filter operating on \( \vec{s} \). We chose \( \vec{g} \), the estimate of \( g \), to be a linear function of the observable \( \vec{z} \); thus \( \vec{g} = \vec{x}^T \cdot \vec{z} \), where \( \vec{x} \) is a vector to be determined.

Under the assumptions that, first, the signal \( \vec{s} \) is generated by a stationary process with known expectation \( \vec{m}_s = \mathbb{E} [\vec{s}] \) and autocorrelation matrix \( \vec{R}_s = \mathbb{E} [\vec{s} \cdot \vec{s}^T] \), and second, that the noise term \( \vec{v} \) is a zero-mean random vector with known covariance matrix \( \vec{C}_v \), uncorrelated with \( \vec{s} \), we take the usual mean-square error \( e = \mathbb{E} [g - \hat{g}] \) as optimization criterion, and a straightforward computation shows that the optimal choice for \( \vec{x} \) is given by

\[
\vec{x} = (\vec{K} \vec{R}_s \vec{K}^T + \vec{R}_v)^{-1} \vec{K} \vec{R}_s \cdot \vec{h}.
\]

Note that this does not at all require any of the involved processes to be Gaussian; this holds for all statements and procedures in the present paper. Correspondingly, the LS-optimal estimate \( \hat{g} \) is given by

\[
\hat{g} = \vec{h}^T \cdot \vec{R}_s \vec{K}^T (\vec{K} \vec{R}_s \vec{K}^T + \vec{R}_v)^{-1} (\vec{K} \vec{s} + \vec{v}).
\]

In the case where the measurement data are noiseless, the previous equation simplifies to the interpolation formula

\[
\hat{g} = \vec{h}^T \cdot \vec{R}_s \vec{K}^T (\vec{K} \vec{R}_s \vec{K}^T)^{-1} \vec{K} \vec{s}.
\]

Finally, if the vector \( \vec{h} \) is successively chosen as the canonical basis vectors

\[
(1,0,0,\ldots,0)^T, (0,1,0,\ldots,0)^T, \ldots, (0,0,0,\ldots,1)^T,
\]

then the optimal interpolating signal \( \hat{\vec{s}} \) for the input vector \( \vec{s} \) is given by

\[
\hat{\vec{s}} = \vec{R}_s \vec{K}^T (\vec{K} \vec{R}_s \vec{K}^T)^{-1} \vec{K} \vec{s}.
\]

It is interesting to note that the vector \( \vec{h} \), which can be thought as a filter applied on \( \vec{s} \), appears in the reconstruction equation (5) only until the last stage. This means that, independent of what \( \vec{h} \) might be, the reconstruction described by eq.(6) always yields the optimum overall result in the least-square sense. Readers familiar with the Gauss-Markov theorem will immediately recognize this as the bare essence of the theorem, which provides the MMSE estimator for all linear functions of the unknown data.
3. ESTIMATION OF AUTOCORRELATION FUNCTION

It is extremely important not to estimate the ACF of the signal $\vec{s}$ using a naive approach. A careless estimation might render an unsymmetric ACF with maximum value at a point different from the origin, in strong disagreement with the theoretical ACF. Furthermore, a strongly biased estimate of the ACF will cause illconditioned behaviour of the inversion term in eq.(5), which in turn produces undesired unstability artifacts (ondulations) on the reconstructed signal. This effect can be avoided by appropriate symmetrizations of the input image data, that is: by estimating the ACF not only on the original image but also on the inverted image and all appropriate reflections of the image.

Examples of estimation results of the ACF of natural images using this procedure are shown in Fig.3. Images of the ocean a) or architecture b) show strong directions of orientation in their corresponding ACFs. High frequency images containing many small independent elements such as c) show a typical fast decay on their ACF, whereas low frequency images such as d) have a smoother ACF with a flattened peak.

4. RECONSTRUCTION RESULTS

In this section we discuss specific details concerning the implementation of the reconstruction formulas derived previously. In case that a pointwise reconstruction is desired, one can use eq.(5); otherwise a blockwise reconstruction is possible through eq.(6). The blockwise reconstruction is much faster, but gives also rise to weak block artifacts which are not to be observed under a pointwise reconstruction.

From now on, the input signal $\vec{s}$ consists of gray-values of an image block appropriately arranged. The matrix $\mathbf{K}$ encodes the information on the observability of the values of $\vec{s}$. For instance, for a $2 \times 2$ image block with the upper right pixel missing, then $\vec{s} \in \mathbb{R}^4$ and $\mathbf{K}$ is a $3 \times 4$ matrix given by

$$
\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

that is, the second row of the $4 \times 4$ identity matrix is removed. Consequently, the size of the block used for the reconstruction must be chosen in such a way that the block has at least one available data point. In order to show the flexibility of the reconstruction formula, we test it on three different scenarios: image extrapolation, image inpainting, and image reconstruction from sparse samples.
**Image extrapolation.** The need for extrapolation occurs when image operators of size $J \times J$ are to be applied to images of size $L \times L$: this either leads to a smaller output image format or requires a reasonable extrapolation of the image. The usual practice of reflecting the image at its borders leads to strong artifacts when this extended signal is filtered. Fig. 2 (a) shows two types of areas to be extrapolated: areas adjacent to image edges (area A) and areas around image corners (area B). To extrapolate in area A, the input vector $\vec{s}$ used is a block containing the nearest available parts of the signal, interpreting the outside of the image as missing data. The reconstructed data consists of an 'outside pixel strip', see Fig. 2 (b) and (d). This layout yields smooth continuous transitions between the extrapolated strips. The extrapolation is performed by shifting the input block and its output strip along the image corner. In the corners, a single block reconstruction is performed, as shown in Fig. 2 (c). Results for three different image extrapolations are shown in Fig. 3.

![Image extrapolation scheme](image extrapolation scheme)

Figure 2. Description of the extrapolation scheme for the edges and corners.

**Reconstruction from sparse samples.** The very same reconstruction method, though with a significantly different geometrical arrangement of known and missing pixels is employed when images are to be reconstructed from sparse samples. Example results are shown in Fig. 6. For these input images of size $255 \times 255$ only 15 percent of the pixel values, placed at random locations, were available. The reconstruction was done pixel by pixel, taking the $15 \times 15$ image block centered at the desired reconstruction location as input vector $\vec{s}$. The block size should be selected to ensure that the blocks contain at least some few valid pixels.

**Image inpainting.** The final application for the reconstruction method shown here is filling in missing values for larger 'holes' of missing data in an image with the goal of preserving the structure present at the available data points. Fig. 4 shows inpainting results for texture images from the Brodatz set degraded by 'holes'. In an interactive application, e.g for retouching purposes, this type of texture inpainting can easily adapt to the local structure by manually delineating the area from which the ACF of the desired texture is to be determined. Reconstructions of images afflicted by a regular pattern of large occlusion blocks are shown in Fig. 5. In all inpainting examples shown here (Fig. 4 and 5), the reconstruction was performed blockwise.

5. CONCLUSION

It has been shown that different tasks such as image inpainting, reconstruction from randomly placed sparse samples, and image extrapolation can be consistently handled under a common statistical framework, based on a Wiener-style reconstruction estimate. The proposed method relies on a simple statistical signal model,
requiring only an estimate of the ACF of both the original and the noise signal. We do not see many practical situations where these very mild requirements cannot be met. Finally, the scheme presented here gives a clear and simple answer on how to extrapolate or interpolate images in those cases where not the image itself, but the results of applying linear operators with extended support (filters), are required. Interestingly, this applies also to spectrum analysis (e.g. FFT of incomplete data). But this is a different story...

REFERENCES

Figure 5. Image inpainting with regular pattern occlusion blocks. On the top left, an input image with occlusions of size $8 \times 8$ and the other three images show reconstructions under this occlusion pattern. The images were reconstructed blockwise using an input mask of size $12 \times 12$.

Figure 6. Four natural images reconstructed from sparse samples (15 percent randomly placed known pixels).


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